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CENTER FOR RADIOPHYSICS AND SPACE RESEARCH
CORNELL UNIVERSITY
ITHACA, NEW YORK

May, 1962

CRSR 117

# MAGNETIC EFFECTS OF GEOMAGNETICALLY TRAPPED PARTICLES

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## ACKNOWLEDGEMENTS

This problem was proposed by Professor T. Gold to whom I am also indebted for helpful suggestions. I am grateful to Professor H. Bondi for valuable discussions. I also wish to express my thanks to Mr. G. Petznick for his assistance during the initial stages of programming for the Burroughs 220 at the Cornell Computing Center and to Dr. J. C. Cain for making available to me the computing facilities of the Goddard Space Flight Center where the final calculations were carried out on an IBM 7090. The work was supported by grants from the Air Force Office of Scientific Research (AF49(638)-915) and from the National Aeronautics and Space Administration (NsG 184-60).

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## I. FORMULATION OF THE PROBLEM

Following solar flare activity there is an enhancement of trapped particles in the geomagnetic field (Van Allen and Lin, 1960) and a simultaneous occurrence of magnetic storms. It was suggested by Singer (1957) that the main phase of magnetic storms might originate in the motion of geomagnetically trapped solar particles, and Dessler and Parker (1959) showed in a semi-qualitative way that trapped particles can possibly account for the average features of storms. Besides the existence of trapped particles in the Van Allen belt, observations have been made which indicate the presence of trapped particles beyond 5 earth radii (Smith, et al., 1960), (Sonnet, et al., 1960).

It is of some interest to be able to calculate the magnetic field produced by the motion of geomagnetically trapped particles. However, the present lack of information about the disposition of trapped particles and the complexity of the earth's field set severe limits on any attempts at a detailed analysis of the magnetic effects of trapped radiation, and therefore, only a relatively simple, mathematically tractable approximation can be attempted. Akasofu and Chapman (1961) calculated the magnetic field disturbance in the equatorial plane for a variety of particle distributions in the trapping region, and Akasofu, Cain, and Chapman (1961) calculated the field everywhere in space for a particular distribution. The discussion which follows describes a means of calculating the field due to an arbitrary distribution of trapped particles at a point which is not in the trapping region.

Consider the following idealized situation. Equal numbers of positively and negatively charged particles are permanently trapped in the field of a magnetic dipole located at the center of a spherical, non-magnetic earth (see Figure 1). We ignore the diamagnetic effects of the earth and macroscopic electric fields in the trapping region. We make no attempt to include the effects of the particle trapping mechanism or the way particles are lost from the trapping region. Since, in fact, particles with mirror points in or below the earth's atmosphere are very quickly lost from the trapping region, we assume that the motion of trapped particles is restricted so that none enter the atmosphere which is assumed to be uniformly 1200 km deep (Johnson, 1960). Moreover, we suppose that the energy density U of the trapped particles is less than the energy density of the trapping field B, i.e., $U \ll B^2/8\pi$. This assumption will justify the use of dipole field equations when analyzing the motion of trapped particles. How the energy is distributed among the individual particles is irrelevant since only the total energy and its spacial distribution are important for calculating the magnetic field. Of course, we cannot have a few particles of such high energy that the adiabatic approximation breaks down or so many low energy particles that collisions become important.

When considering the motion of an individual particle, we assume that its instantaneous center of gyration always remains on the same dipole shell, thereby disregarding the effects of particle motion from shell to shell. The adiabatic invariance of the magnetic moments of the particles is taken for granted throughout the discussion.

In attempting to calculate the magnetic effects of trapped particles, we can resolve the motion of the particles into several components by virtue of the adiabatic approximation and look at the field implied by each. Firstly, a trapped particle moves in a helical path along a field line, the pitch of the helix growing flatter with increasing field strength (see Figure 1). At some point the field becomes so strong that the particle loses its component of motion along the field line and then "mirrors", i.e., starts back toward the region of weakest field. It continues beyond the point of weakest field until reflection occurs again. The helical trajectory can be further analyzed into a circular motion around the particle's guiding center (i.e., the instantaneous center of gyration) and the oscillation of the guiding center between mirror points. At the same time, as a result of the curvature of the field lines and the inhomogeneity of the field, the particles drift around the earth, positive charges moving westward and negative charges moving eastward. The oscillatory motion has no net magnetic effect. On the other hand, the drift of the particles creates a ring current around the earth while the gyration motion results in a magnetic moment per unit volume in the trapping region. It is to these two sources of the magnetic disturbance field that we shall direct our attention.

Before beginning the analysis, a few remarks about notation are in order. We shall use B to denote the trapping field and H to indicate the field produced by the motion of the trapped particles. When discussing the disturbance field, we also adopt the convention that primed coordinates refer to source points while unprimed coordinates refer to field points, unless otherwise stated. Thus field points of the trapping field are primed since they are source points of the trapped particle field. In addition, the subscripts D and G refer to drift current field and gyration field quantities respectively. Note also that, unless otherwise stated, physical quantities are measured in Gaussian units.

# II. ANALYSIS

## A. Basic Equations

We begin by presenting some of the geometric properties of a dipole field as well as relevant equations of motion of the trapped particles. Although the region in which charged particles are trapped is originally a dipole field, the eventual presence of trapped particles guarantees that the field will be distorted. However, we assumed that $U << B^2/8\pi$ (or equivalently that the field is not distorted much), and so use the equations for an unperturbed dipole field. The distortion of the field at the earth's surface is, in fact, never very large.

### 1. The Geometry of a Dipole Field

In spherical polar coordinates the potential due to a magnetic dipole with its north pole points in the negative z direction is

$$V = -\frac{M\cos\theta'}{r'^2} \tag{1}$$

where M is the magnetic moment of the dipole, $\theta'$ is the colatitude of the field point, and $r'$ is the distance to the field line. The resulting axially symmetric field has components

$$B_{r'} = -\frac{2M\cos\theta'}{r'^3} \tag{2}$$

$$B_{t'} = -\frac{M\sin\theta'}{r'^3} \tag{3}$$

and magnitude

$$B = |\underline{B}| = B_e\frac{(1+3\cos^2\theta')^{1/2}}{\sin^6\theta'} \tag{4}$$

$B_e$ being the equatorial value of magnetic induction on the field line in question.

The equation of a dipole field line is

$$r' = a \sin^2\theta' \tag{5}$$

where the parameter, a, represents the equatorial distance from the dipole to the field line. Hence, the differential element of arc length along a line of force is

$$ds = a \sin\theta' (1 + 3\cos^2\theta')^{1/2} d\theta' \tag{6}$$

and its radius of curvature is

$$R = \frac{1}{3} a \sin\theta' \frac{(1 + 3\cos^2\theta')^{3/2}}{(1 + \cos^2\theta')} \tag{7}$$

Equations (3) and (4) can be used to write B in terms of $B_o$, the equatorial value of the field at one earth radius, vis.,

$$B = B_0 \frac{R_e^3}{a^3} \frac{(1 + 3\cos^2\theta')^{1/2}}{\sin^6\theta'} \tag{8}$$

where $R_e$ is the radius of the earth in centimeters.

2. <u>The Motion of Charged Particles in an Inhomogeneous Magnetic Field</u>

Much has been written about the equations of motion of a charged particle in external fields (e.g., see Alfven (1950)), and a concise review of the subject is given by Spitzer (1956). Therefore only pertinent results will be included here.

A particle of mass m and charge q in a magnetic field B has a cyclotron frequency

$$\omega_c = \frac{qB}{mc} \tag{9}$$

where q is measured in electrostatic units. Under the influence of external forces, particles in a magnetic field will drift in a direction perpendicular to both the magnetic field and the external force. In the case of the geomagnetic field, particle drift is primarily due to the inhomogeneity of the field along equipotentials and to the centrifugal force the particles experience as they move along the curved lines of force. The drift velocity due to the field gradient is

$$W_{Dg} = \frac{\delta \nabla_\perp B}{2B} w_\perp \tag{10}$$

and the drift velocity due to the curvature of the lines of force is

$$W_{Dc} = \frac{w_{\parallel}^2}{R\omega_c} \tag{11}$$

where $\delta$ is the radius of gyration of the particle, and $\omega_c$ is its cyclotron frequency. $w_\perp$ and $w_\parallel$ are respectively the perpendicular and parallel components of the particle's velocity $\underline{w}$ with respect to the field. Equation (10) is applicable only if $\nabla_\perp B$, the gradient of the scalar B normal to $\underline{B}$, is small compared to B. In the interest of simplicity we assume that $\nabla \times \underline{B} \sim 0$ in the trapping region (i.e., the distortion of the trapping region is small), in which case the total drift velocity can be written as

$$W_D = W_{Dc} + W_{Dg} = \frac{1}{R\omega_c}\left(\frac{1}{2} w_\perp^2 + w_\parallel^2\right) \tag{12}$$

Let $\alpha$ be the pitch angle of the particle, i.e., the angle between the instantaneous velocity vector of the particle and the local magnetic field. If we define u as the kinetic energy of the particle ($u = 1/2\ mw^2$) and set

$$u_D = \left(\frac{1}{2}\sin^2\alpha + \cos^2\alpha\right) u \tag{13}$$

then Equation (12) for the drift velocity becomes

$$W_D = \frac{2c}{\varrho BR} u_D \tag{14}$$

where

$$W = |\underline{W}| \tag{15}$$

Using (7) and (8) we arrive at the expression

$$W_D = \frac{6ca^2}{\varrho B_0 R_2^3} \frac{\sin^5\theta'(1+\cos^2\theta')}{(1+3\cos^2\theta')^2} u_D \tag{16}$$

One more relation we shall find useful follows immediately from the adiabatic invariance of the magnetic moment

$$\mu = u \frac{\sin^2 \alpha}{B} \tag{17}$$

and the assumption that the kinetic energy of the particles is constant, vis.,

$$\frac{\sin^2 \alpha}{B} = const. \tag{18}$$

B. Magnetic Effects of Trapped Particles

As we mentioned earlier, the main forces the trapped particles experience arise from (i) the motion of the particles normal to the magnetic field, (ii) the inhomogeneity of the field along equipotentials, and (iii) the curvature of the field lines. The magnetic effect of (i) is calculated by introducing a scalar potential for the magnetic field outside the trapping region written in terms of an intensity of magnetization. The forces due to (ii) and (iii) cause the particles to drift around the geomagnetic axis, and the magnetic field of the resulting ring current is also described in terms of a scalar potential. The magnetic effect of the drift motion is examined first.

1. The Drift Current Field

If we view the drift motion of the particles as a current in the trapping region, then the charge density is proportional to the trapped particle density. Both the particle density and drift velocity are functions of the energies and local pitch angles of the trapped particles. Since the drift velocity in the axially symmetric magnetic field of the earth is independent of the azimuthal angle $\phi$, the drift current can be treated as a continuous distribution of circular current loops which lie in planes perpendicular to the geomagnetic axis with their centers on the axis. The direction of the current is westward around the earth.

a. The Scalar Magnetic Potential of a Circular Current Loop

Let $d\Omega_{DA}$ be the magnetic scalar potential at the point A on the axis of a loop of current density $j(r', \theta')$. Referring to Figure (2) we let h be the distance from the origin O to the field point A. In terms of the ordinary Legendre polynomials $P_n(\cos\theta')$,

$$d\Omega_{DA} = 2\pi j(r', \theta') \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{r'}{h}\right)^{n+1} \sin^2\theta' \frac{d P_n(\cos\theta')}{d(\cos\theta')} d\Sigma \tag{19}$$

where $d\Sigma$ is the cross-sectional area of the current loop. For fixed r' and θ' the potential at an arbitrary field point $P(r, \theta, \phi)$ is therefore

$$d\Omega_D(\underline{r}, r', \theta') = 2\pi j(r', \theta')(1 - \cos\theta') d\Sigma - 2\pi j(r', \theta') \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r'}\right)^{n} \sin\theta' P_n^1(\cos\theta') P_n(\cos\theta) d\Sigma \tag{20}$$

when $|\underline{r}| < r'$ (Ferraro, 1954).

It proves convenient to use the quantities "a" and θ' rather than r' and θ' as the independent variables, in which case (20) becomes

$$d\Omega_D(\underline{r}, a, \theta') = 2\pi j(a, \theta')(1 - \cos\theta') d\Sigma - 2\pi j(a, \theta') \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^{n} \frac{P_n^1(\cos\theta')}{\sin^{2n-1}\theta'} P_n(\cos\theta) d\Sigma \tag{21}$$

In order to calculate the current density of particles $j(a, \theta')$ we must consider how particles are distributed in the trapping region.

b. The Particle Distribution

Swann (1933) has shown that for a system of N charged particles in the presence of an external magnetic field, the 6N dimensional phase space density is constant along the trajectory of a system point in phase space. If the particles are mutually non-interacting, then the Hamiltonian of the system can be written as the sum of the Hamiltonians of the individual particles, and this implies that the phase space density in a 6 dimensional phase space is constant along the trajectory of the phase point of a particle. Moreover, if the kinetic energy of each particle is constant, then the directional differential intensity in ordinary configuration space is constant. Since we have in fact assumed that collisions are rare and $\nabla \times \underline{B}$ is small in the trapping region, the above result is applicable to the present problem and provides a means of inferring the particle density everywhere on a dipole shell from the equatorial particle energy density and pitch angle distribution.

Let dS be an area oriented so that its normal is in the direction of motion of the particle. Then the directional differential intensity I is the number of particles per unit area per unit energy per unit time per unit solid angle (centered around the direction of motion of the particle), i.e.,

$$I = \frac{dN}{dS\, d\omega\, dE\, dt} = \text{constant} \tag{22}$$

The solid angle $d\omega$ is measured in terms of the azimuthal angle $\psi$ of the particle around the local field line and the pitch angle $\alpha$.

Moreover, the number of particles per unit volume per unit solid angle per unit energy at the point (a, $\theta'$) is then

$$\rho(a,\omega,E) = \frac{I(a,\omega,E)}{w} \tag{23}$$

I and $\rho$ are implicitly functions of $\theta'$ since $\alpha = \alpha(\theta')$. Consequently, $\rho$ is constant in the neighborhood of a particle as it moves through the trapping region. Note that this does not say that the particle density integrated over all pitch angles and all energies is constant thoughout space. This will be true only if the pitch angle distribution is isotropic. What is always the case, however, is that we can calculate the total particle density at every point on the dipole shell from the value of the integrated $\rho$ in the equatorial plane.

We shall asumme that I (a, ω, E), and therefore $\rho$ (a, ω, E), are independent of the azimuthal angle $\psi$. In general I and $\rho$ vary with pitch angle $\alpha$. Parker (1957) suggested a simple class of distribution functions which in our notation are of the form $I(a,\omega,E) \propto (B_e/B)^{(f-1)/2} \sin^{f-1}\alpha$. If we assume that f = 1, then we have a velocity distribution which deviates from isotropy only to the extent that particles which fail to satisfy the mirror point restriction set by the atmosphere are absent. This distribution shall be referred to as modified isotropy.

c. <u>The Drift Current Density j (a, θ')</u>

The differential ring current density around the earth resulting from the drift motion of trapped particles is

$$dj(a,\theta',\omega,E) = \rho(a,\omega,E)\,\frac{q}{c}\,w_D\,dE\,d\omega \tag{24}$$

where q is the charge of the particles in esu, and c is the speed of light. On substituting Equation (16) for the drift velocity into the above expression and integrating over all energies and all allowed pitch angles we get

$$j(a,\theta') = \frac{6}{B_0}\,\frac{a^2}{R_e^3}\,\frac{\sin^5\theta'(1+\cos^2\theta')}{(1+3\cos^2\theta')^2}\,U_D \tag{25}$$

where

$$U_D = \int dE \iint \rho(a,\omega,E)\,u_D\,d\omega \tag{26}$$

and $u_D$ is given by Equation (13).

To digress for a moment, $U_D$ can be written in terms of the energy density U of the trapped particles. In general

$$U(a,\theta') = \int E\,dE \iint \rho(a,\omega,E)\,d\omega \tag{27}$$

and assuming modified isotropy this becomes

$$U(a,\theta') = 4\pi\cos\alpha_{min}\,\beta(a) \tag{28}$$

where

$$\beta(a) = \int E\, \rho(a, E)\, dE \tag{29}$$

and $\alpha_{min}$ is the minimum pitch angle a particle can have consistent with the requirement that the implied mirror point be above the atmosphere. It is convenient to write $U(a,\theta')$ in terms of $U^* = U(a_o, \pi/2)$, the equatorial particle energy density on the field line specified by $a_o$. Thus

$$U(a,\theta') = \frac{4\pi \cos\alpha_{min}(a,\theta')\, \beta(a)}{4\pi \cos\alpha_{min}(a_o, \pi/2)\, \beta(a_o)} U^*(a_o, \pi/2) \tag{30}$$

In Figure (3) $\cos \alpha_{min}$ $(\pi/2, a)$ is plotted as a function of "a". We rewrite Equation (30) as

$$U(a,\theta') = \cos\alpha_{min}(a,\theta')\, \frac{\bar\beta(a, a_o)}{\gamma(a_o)}\, U^*(a_o, \pi/2) \tag{31}$$

where

$$\bar\beta(a,a_o) = \frac{\beta(a)}{\beta(a_o)}$$

and

$$\gamma(a_o) = \cos\alpha_{min}(a_o, \pi/2) \tag{32}$$

To return now to the main line of argument, integrating Equation (26) over $\alpha$ and $\psi$, we then get

$$U_D(a,\theta') = \frac{1}{2}\left(1 + \frac{1}{3}\cos^2\alpha_{min}\right) U(a,\theta') \tag{33}$$

Finally, we calculate $\cos \alpha_{min}$.

Let $r_{atm}$ be the distance from the center of the earth to the level of the atmosphere above which particles can mirror. From (5), the minimum colatitude accessible to a particle is

$$\eta(a) = \sin^{-1}(r_{atm}/a)^{1/2} \tag{34}$$

Using (8), (18), and (34), we then have

$$\cos\alpha_{min} = \left[1 - \frac{(1+3\cos^2\theta')^{1/2}}{\sin^6\theta'}\, \frac{r_{atm}^3}{a^2(4a^2 - 3a\, r_{atm})^{1/2}}\right]^{1/2} \tag{35}$$

d. The Magnetic Field Due to the Drift Motion of Trapped Particles

Let

$$b_n = 2\pi j(a,\theta')\frac{1}{n}\left(\frac{1}{a}\right)^n \frac{P_n^1(\cos\theta')}{\sin^{2n-1}\theta'} \tag{36}$$

Then the potential due to the drift motion of the particles given by Equation (21) becomes

$$d\Omega_D(r,a,\theta') = 2\pi j(a,\theta')(1-\cos\theta')\,d\Sigma - \sum_{n=1}^{\infty} b_n r^n P_n(\cos\theta)\,d\Sigma \tag{37}$$

Integrating over the entire trapping region gives the magnetic potential due to all the trapped particles

$$\Omega_D(r,\theta) = \int_{a_1}^{a_2} da \int_{\eta(a)}^{\pi-\eta(a)} \Big[2\pi j(a,\theta')(1-\cos\theta') - \sum_{n=1}^{\infty} b_n r^n P_n(\cos\theta)\Big]\frac{|J(a,\theta',\phi')|}{a\sin^3\theta'}\,d\theta' \tag{38}$$

where J (a,θ',ø') is the Jacobian determinant.

Since

$$\underline{H} = -\nabla\Omega_D \tag{39}$$

the components of the magnetic field for $r < r_{atm}$ arising from the drift motion of the trapped particles are

$$H_r = \int_{a_1}^{a_2} da \int_{\eta(a)}^{\pi-\eta(a)} \Big[\sum_{n=1}^{\infty} n\, b_n r^{n-1} P_n(\cos\theta)\Big]\frac{|J(a,\theta',\phi')|}{a\sin^3\theta'}\,d\theta' \tag{40}$$

and

$$H_t = \int_{a_1}^{a_2} da \int_{\eta(a)}^{\pi-\eta(a)} \Big[\sum_{n=1}^{\infty} b_n r^{n-1} \frac{d P_n(\cos\theta)}{d\theta}\Big]\frac{|J(a,\theta',\phi')|}{a\sin^3\theta'}\,d\theta' \tag{41}$$

If we measure "a" and r in units of earth radii and use Equation (25) for the current density, the drift field components become

$$H_r = \int_{a_1}^{a_2} da \int_{\eta(a)}^{\pi-\eta(a)} \left[ \sum_{n=1}^{\infty} n\, b_n\, r^{n-1} P_n(\cos\theta) \right] d\theta' \tag{42}$$

and

$$H_t = -\int_{a_1}^{a_2} da \int_{\eta(a)}^{\pi-\eta(a)} \left[ \sum_{n=1}^{\infty} b_n\, r^{n-1} P_n^1(\cos\theta) \right] d\theta' \tag{43}$$

where

$$b_n = -\frac{12\pi}{B_0}\frac{1}{n}\, a^3 \sin^9\theta' \,\frac{(1+\cos^2\theta')}{(1+3\cos^2\theta')}\,\frac{P_n^1(\cos\theta')}{(a\sin^2\theta')^{n-1}}\, U_D \tag{44}$$

The evaluation of (42) and (43) is considered in Appendix A.

2. The Gyration Field

The gyration of trapped particles around lines of force results in a net magnetic moment per unit volume in the trapping region. The potential due to the alignment of the axes of gyration of the particles antiparallel to the earth's field is obtained from the solution of Poisson's equation in a source free region. After deriving a formal expression for the potential due to the above magnetic sources, we calculate the intensity of magnetization in the trapping region and the equations for the magnetic field components.

a. The Scalar Magnetic Potential of a Region of Magnetic Polarization

If $\underline{M}$ is the intensity of magnetization in the trapping region, then the equivalent magnetic source density is $\nabla \cdot \underline{M}$ and Poisson's equation becomes

$$\nabla^2 \Omega_G = 4\pi\, \nabla \cdot \underline{M} \tag{45}$$

The solution of (45) valid inside a sphere of radius $r = r_{atm}$, which is a source free region, is

$$\Omega_G(r,\theta,\phi) = \sum_{n=0}^{\infty} b_n\, r^n P_n(\cos\theta) \tag{46}$$

where

$$b_n = 2\pi \int_0^{\pi} d\theta' \int_{r_{atm}}^{\infty} \frac{\nabla \cdot \underline{M}\, P_n(\cos\theta')}{r'^{(n-1)}} \sin\theta' \, dr' \tag{47}$$

Equation (46) is derived in Appendix B.

b. Magnetization of the Trapping Region

The adiabatic invariance of the magnetic moments of the particles permits us to write

$$\rho_G(r', \theta') = \int_0^{\infty} dE \iint \rho(r', \theta', \omega, E)\, d\omega \tag{48}$$

as an approximate expression for the guiding center density $\rho_G$. Since each particle has magnetic moment

$$\mu = \frac{u \sin^2\alpha}{B} = \frac{u_G}{B} \tag{49}$$

the magnetic moment per unit volume is

$$|\underline{M}(r', \theta')| = \frac{1}{B} \int_0^{\infty} dE \int_0^{2\pi} d\psi \int_{\alpha_{min}}^{\pi - \alpha_{min}} u_G\, \rho(r', \theta', \omega, E) \sin\alpha \, d\alpha \tag{50}$$

If we assume modified isotropy, then

$$|\underline{M}(r', \theta')| = \frac{U_G}{B} \tag{51}$$

where

$$U_G = \frac{1}{2}\left(1 - \frac{1}{3}\cos^2\alpha_{min}\right) u \tag{52}$$

Further, the direction of the intensity of magnetization is anti-parallel to the direction of the local trapping field, and so

$$\underline{M} = -\frac{\underline{B}}{B} |\underline{M}| \tag{53}$$

and

$$\nabla \cdot \underline{M} = -\frac{\underline{B}}{B^2} \cdot \nabla U_c - U_c \nabla \cdot \frac{\underline{B}}{B^2} \tag{54}$$

If we let L = cos $\alpha_{min}$, then the divergence of the intensity of magnetization is

$$\nabla \cdot \underline{M} = \frac{2\pi\gamma U^*}{3 B_0} \frac{r'^2 L}{(1+3\cos^2\theta')} \Big\{ (3-L^2)\Big(2r' \frac{\partial \bar{\beta}}{\partial r'} \cos\theta' + \frac{\partial \bar{\beta}}{\partial \theta'} \sin\theta'\Big) + \frac{3\cos\theta' \bar{\beta}(3+5\cos^2\theta')}{2L^2(1+3\cos^2\theta')} (3+6L^2-L^4) \Big\} \tag{55}$$

c. The Magnetic Field Due to the Gyration of Trapped Particles Around Lines of Force

The components of the gyration field are easily obtained from Equation (46) for the potential. They are

$$H_r = \sum_{n=0}^{\infty} n b_n r^{n-1} P_n(\cos\theta) \tag{56}$$

and

$$H_t = -\sum_{n=0}^{\infty} b_n r^{n-1} P_n^1(\cos\theta) \tag{57}$$

Let

$$G(r', \theta') = \frac{3 B_0}{2\pi\gamma U^*} \nabla \cdot \underline{M} \sin\theta' \tag{58}$$

Then Equation (47) becomes

$$b_n = \frac{4\pi^2 \gamma U^*}{3 B_0} \int_0^{\pi} d\theta' \int_{r_{atm}}^{\infty} \frac{G(r', \theta')}{r'^{(n-1)}} P_n(\cos\theta') \, dr' \tag{59}$$

The evaluation of (56) and (57) is considered in Appendix A.

3. Approximating the Field Components at the Earth's Surface

A satisfactory estimate of the field components at the earth's surface can be obtained with a first order calculation assuming a centered dipole trapping field (Akasofu, Cain, and Chapman, 1961 ). However, this involves a lengthy numerical solution of the field equations. In view of our limited knowledge of the structure of the magnetosphere, still less precise estimates of the field produced by trapped particles can often provide equally useful information. As we shall show, an order of magnitude calculation of the field implied by an arbitrary particle energy density distribution is readily inferred from the field implied by a particle energy density distribution which is uniform in the equatorial plane and which varies along the field lines in accordance with a modified isotropic pitch angle distribution.

In practice the calculation depends on two factors. First we must know the field components due to narrow belts of trapped particles with a uniform equatorial particle energy density distribution and a modified isotropic pitch angle distribution. Actually it is better to deal with the dimensionless quantities $I_t(\theta) = H_t B_o/\gamma U^*$ and $I_r(\theta) = H_r B_o/\gamma U^*$. $H_t$ and $H_r$ are respectively the transverse and radial components of the field at colatitude $\theta$ produced by the trapped particles, $B_o$ is the equatorial value of the earth's field at the surface of the earth, $U^*$ is the equatorial particle energy density of the dipole shell, and $\gamma$ is the cosine of the equatorial pitch angle of a particle which mirrors at the top of the atmosphere. Ordinarily $\gamma = 1$ as Figure 3 shows. Computer calculated values of $I_r$ and $I_t$ at $10^o$ intervals of latitude are tabulated in Tables 1a and 1b for belts which are 0.1 $R_e$ wide in the equatorial plane, where $R_e$ is the radius of the earth. The range of "a" (the equatorial radius of the inner field line measured in earth radii) in the tables is $1.3 \leq a \leq 4.0$ and the entries are accurate to 1%. Because the field assumes a particularly simple form when $a > 4$, the tabulation ends at this point. In Figures 4 and 5 $I_r$ and $I_t$ are plotted for $1.3 \leq a \leq 2.5$ in steps of 0.2. We have assumed in obtaining these results that none of the trapped particles has a mirror point in or below the atmosphere whose upper limit is 1200 km above the surface of the earth.

The second consideration in the order of magnitude calculation is the equatorial particle energy density distribution of interest. This is taken into account through the relative particle energy density distribution, $\bar{\beta}(a,a_o)$. At $a = a_o$, $\bar{\beta}(a,a_o) = 1$ and the value of $U^*$ at $a_o$ serves to specify $U^*$ everywhere in the equatorial plane. Moreover, the value of $\gamma$ becomes $\gamma(a_o)$ for all "a" when $\bar{\beta} = \bar{\beta}(a, a_o)$. We shall now consider the approximation in greater detail and consider the errors involved. Finally, we shall give an example of how the approximation procedure is used.

The discussion which follows will refer to the radial component of the field produced by trapped particles although it applies without change to the transverse component. The radial component can be expressed formally as

$$H_\lambda(\theta) = \int_\lambda^\mu \bar{\beta}(a,a_o)\, da \int F_\lambda(a,\theta,\theta')\, d\theta' \tag{60}$$

The $\theta'$ limits of integration, which have been suppressed to simplify notation, are functions of "a". If the particle energy density is uniform in the equatorial plane, we can write

$$H_\lambda(\theta) = \sum_{i=1}^{N} \int_{\lambda+(i-1)\Delta a}^{\lambda+i\Delta a} da \int F_\lambda(a,\theta,\theta')\, d\theta' = \sum_{i=1}^{N} \frac{\gamma U^*}{B_o} I_{\lambda_i}(\theta) \tag{61}$$

where $\mu = \lambda + N\Delta a$. Equation (61) represents the radial component of the magnetic field at the surface of the earth due to a series of N belts, each of width $\Delta a$ in the equatorial plane with $\bar{\beta}(a, a_o) = 1$.

Now to estimate $H_r(\theta)$ for an arbitrary $\bar{\beta}(a, a_o)$ on the interval $(\lambda,\mu)$, we multiply each integral $I_{r_i}$ by any value of $\bar{\beta}(a, a_o)$ on the interval $(\lambda + (i-1)\ \ a, \lambda + i\ \ a)$; thus

$$H_\lambda(\theta) \approx \frac{\gamma(a_o)\, U^*(a_o, \pi/2)}{B^o} \sum_{i=1}^{N} \bar{\beta}(a_i, a_o)\, I_{\lambda_i}(\theta) \tag{62}$$

Some simplification is possible since particles trapped on dipole shells for which $a \geq 4$ produce a uniform field inside the earth to an accuracy of 1%; even when $2.5 \leq a \leq 4$, the deviation from uniformity is less than 10%. This effect will be discussed in greater detail below. We can therefore write Equation (62) as two sums; one represents the contribution of particles trapped closer than, say, $a = 2.5$ in the equatorial plane, and the other represents the contribution of particles trapped beyond $a = 2.5$.

A further simplification is possible in the case of the distant particle contribution. That the magnetic field throughout the earth due to trapped particles is uniform implies that only $P_1$ (cos θ) and $P_1^1$ (cos θ) are important, i.e., the field falls off like 1/a. Since $H_r \propto J/a$ where J is current, and the current density $j \propto \frac{1}{R\,B}$ where R is the radius of curvature of the field line, we have $H_r \propto$ "a" times the area between dipoles shell. Since the area between dipole shells is proportional to $a \Delta a$, and $\Delta a$ is constant, we have $H_r \propto a^2$. When the proportionality constant is calculated, we have

$$H_r(\theta) = \frac{U^*(a_0, \pi/2)}{\gamma(a_0)\, B_0} \sum_{i=1}^{m} \bar{\beta}(a, a_0)\, I_{r_i}(\theta) + 7.6\, \frac{U^*(a_0, \pi/2)}{\gamma(a_0)\, B_0} \cos\theta \sum_{i=1}^{n} \bar{\beta}(a, a_0)\, a_i^2 \qquad (63a)$$

$$H_t(\theta) = \frac{U^*(a_0, \pi/2)}{\gamma(a_0)\, B_0} \sum_{i=1}^{m} \bar{\beta}(a, a_0)\, I_{t_i}(\theta) + 7.6\, \frac{U^*(a_0, \pi/2)}{\gamma(a_0)\, B_0} \sin\theta \sum_{i=1}^{n} \bar{\beta}(a, a_0)\, a_i^2 \qquad (63b)$$

where θ is the colatitude of the field point.

The particle energy density distribution is completely specified by the pitch angle distribution and $\bar{\beta}$ (a, $a_0$); hence, these are the only factors we have to consider in an error analysis. The field implied by trapped particles is relatively insensitive to the pitch angle distribution. Since the field is proportional to $\sin^9 \theta'$ where θ' is the colatitude of the source point, regardless of the pitch angle distribution, the major contribution to the field integrals comes from the motion of the particles near the equatorial plane. The difference between the field due to a modified isotropic distribution and the most extreme deviation from modified isotropy is not greater than a factor of 3. By way of example, Akasofu, Cain and Chapman (1961) calculated the field at the surface of the earth due to particles trapped on dipole shells for which $5 \leq a \leq 7$ with a Gaussian particle energy density distribution in the equatorial plane and a pitch angle distribution which is proportional to $\sin^{1/2} \alpha$, a distribution which emphasizes the role of steep pitched particles. Our calculation for a particle energy density distribution which differs from theirs only to the extent that we use a modified isotropic pitch angle distribution gives a value for H which is one half of their value.

The limitations on $\bar{\beta}$ (a, $a_o$) are intimately related to the accuracy of the estimate and to the width of the belt. If we let $\beta$ be the absolute value of the maximum logarithmic derivative of $\bar{\beta}$ (a, $a_o$) on an equatorial interval of length L, then the error $\epsilon$ in the estimates of $H_r$ and $H_t$, assuming modified isotropy, satisfies the relation

$$L\beta \leq \epsilon \qquad (64)$$

This result follows from the mean value theorems of calculus. The slower $\bar{\beta}$ (a, $a_o$) changes with "a", the better the estimate. Clearly, the total error made in the estimate can never be less than the 1% error inherent in the tabulated values of $I_r$ and $I_t$.

As an illustration of the approximation scheme, consider a Gaussian distribution $\bar{\beta}$ (a, 2.2) = $\exp(-4\,(a - 2.2)^2)$, on the interval $1.6 \leq a \leq 3.4$ with a modified isotropic pitch angle distribution; $\beta \approx 4$ and $\epsilon = 0.7$ (70%). We wish to calculate $H_r$ at colatitude $\theta = 30^\circ$. From $\bar{\beta}$ (a, 2.2) and Table 1a we obtain the first sum in Equation (63a) for $1.6 \leq a \leq 2.5$, vis.,

$$197.5 \, \frac{U^*(2.2, \pi/2)}{\gamma(2.2)\; B_0}$$

The second sum in (63a) is

$$81.0 \cos 30^\circ \frac{U^*(2.2, \pi/2)}{\gamma(2.2)\ B_0} = 70.0 \frac{U^*(2.2, \pi/2)}{\gamma(2.2)\ B_0}$$

From Figure 3 we see that $\gamma = 0.96$ and taking $B_o$ to be 0.31 gauss we have

$$H_r\ (30^\circ) = 9.3 \times 10^2\ U^*\ (2.2)\ \text{gauss}$$

where $U^*$ (2.2) is measured in ergs/cm$^3$. The computer calculated value is $H_r = 8.3 \times 10^2\ U^*$ (2.2) gauss and clearly $\epsilon < 0.7$.

## III. CONCLUSIONS

The analysis we have presented and the calculations of the fields at the surface of the earth due to narrow belts of trapped particles provide a basis for discussing some of the magnetic effects of geomagnetically trapped particles.

The assumption that the particles are trapped in a centered dipole field guarantees the symmetry of magnetic sources about the geomagnetic equatorial plane as well as about the geomagnetic axis, and therefore the same symmetries obtain for the magnetic field at the surface of the earth due to the motion of the particles. Hence, we only have to consider the field along an arbitrary meridian in one hemisphere. The symmetries of the magnetic source are exactly those of sets of coaxial Helmholtz coils.

The axial symmetry permits us to use zonal harmonics and the equatorial symmetry implies that the even harmonics $P_o$, $P_2$, ... contribute nothing to the field. The number of odd harmonics needed to represent the field may be found by expanding the source factors in the drift and gyration integrals in powers of $\cos^2 \theta'$ and using the orthogonality properties of $P_n$ $(\cos \theta')$ and $P_n^1$ $(\cos \theta')$. The same information can be obtained from Stieltjes' bounds for ordinary and associated Legendre polynomials (Sansone, 1959). We conclude that in general it is not necessary to go beyond $P_7$ $(\cos \theta')$ and $P_7^1$ $(\cos \theta')$. The number of terms needed in any particular particle distribution depends on where in the field the particles are trapped and the accuracy desired. All of our calculations include $P_7$ $(\cos \theta')$ and $P_7^1$ $(\cos \theta')$, and the results are accurate to better than 1%.

Figures 4 and 5 illustrate the importance of the various harmonics for belts which are 0.1 earth radius thick in the equatorial plane and all of which have the same particle energy density distribution. We have assumed in the calculations that no particles mirror in the atmosphere whose upper limit is 1200 km above the surface of the earth. Different pitch angle distributions will cause relatively small changes in the latitude dependence of the field components as discussed earlier.

The function $I_t$ plotted in Figure (5) is proportional to $H_t$, the transverse (horizontal) component of magnetic field at the surface of the earth due to trapped particles. The most striking feature of the curves is the general tendency for the transverse (horizontal) component to increase with increasing colatitude, which points to the importance of the $P_1^1(\cos\theta) = \sin\theta$ term. A significant deviation from this rule occurs at a = 1.7. At large colatitudes (i.e., low latitudes) the field is approximately constant and the latitude dependence is very sensitive to "a" around a = 1.7. The constancy of $H_t$ around a = 1.7 will be of importance when we discuss magnetic storms. As "a" becomes larger, the latitude dependence becomes more nearly sinusoidal. In fact, for $a \gtrsim 2.5$ the deviation from a sinusoid is less than 10% and for $a \gtrsim 4.0$ the deviation is less than 1%.

The function $I_r$ plotted in Figure (4) is proportional to $H_r$, the radial (vertical) component of magnetic field at the surface of the earth due to trapped particles. The radial (vertical) component in general decreases with increasing colatitude, which points to the importance of the $P_1(\cos\theta) = \cos\theta$ term. The extent to which $H_r$ deviates from a purely $\cos\theta$ dependence closely parallels the transverse case for $a \gtrsim 2.5$.

For "a" sufficiently large (i.e., $a \gtrsim 2.5$) we have $H_r \propto \cos\theta$ and $H_t \propto \sin\theta$; in other words, $H = \left(H_r^2 + H_t^2\right)^{1/2}$ = constant on the surface of the earth. Now if the field inside the earth due to trapped particles is uniform, then the field on the surface would be uniform. Since the solution of Laplace's equation is unique (up to an arbitrary constant), the existence of a uniform field on the surface of the earth implies a uniform field throughout the earth due to the trapped particles. Akasofu, Cain, and Chapman (1961) came to the same conclusion for the field due to particles trapped on dipole shells for which $5 \leq a \leq 7$.

Since the sources can be viewed as sets of coaxial Helmholtz coils, uniformity of the field inside the earth implies that the radius of the earth is small compared to the radius of the equivalent Helmholtz coils which make the greatest contribution to the field. That this is, in fact, the case shows up clearly in the expressions for the magnetic field components which are proportional to $\sin^9 \theta'$ for a given value of "a" indicating that particle motion near the equatorial plane is the principal source of magnetic field.

The magnetic field throughout the earth due to geomagnetically trapped particles can be calculated from measured values of the particle energy density and pitch angle distributions. A zonal harmonic analysis of the magnetic field at the surface of the earth in conjunction with the calculated trapped particle field gives information about other external sources such as ionospheric currents and high latitude phenomena as well as information about induced currents inside the earth.

The zonal harmonic analysis, which in practice is difficult to carry through, has been done for magnetic storm data (Slaucitajs and McNish, 1936), though only to a rough approximation. Furthermore, information about the disposition of the trapped particles is rather limited, especially for low energy particles (i.e., a few kev or less). Rocket measurements in the auroral zone indicate the presence of large fluxes of 5 kev electrons (McIlwain, 1960) and it is not unreasonable to suppose that such low energy particles make up a large part of the trapped particle flux. So far, measurements in the equatorial region of the geomagnetic field have been confined to electrons with energies greater than 20 kev (O'Brien, Van Allen, Laughlin, and Frank, 1962). Thus the analysis suggested is impractical at present. However, what we can do is essentially the converse, i.e., estimate the particle energy density in the trapping region on the basis of magnetic storm data.

We shall first try to establish where in the earth's field the maximum concentration of particles occurs during magnetic storms. A lower limit of $a = 1.2$ (1200 km above the surface of the earth) is easily established since below this altitude particle density increases very rapidly (Johnson, 1960), and significant concentrations of charged particles have been observed only down to about 1000km (Van Allen, Ludwig, Ray and McIlwain, 1958). An upper limit of $a \lesssim 4$ can be inferred from cosmic ray cutoff experiments (Kellogg and Winckler, 1961). Another line of evidence to justify a still lower upper limit is the presence of harmonics higher than the first in magnetic storm data obtained in low and middle latitudes (Slaucitajs and McNish, 1936), (Chapman and Bartels, 1940). The $P_1$ (cos $\theta$) and $P_1^1$ (cos $\theta$) terms in the radial and transverse components respectively dominate always, although, as Chapman and Bartels point out, the higher harmonics disappear for all practical purposes after the first 10 hours or so of the storm. In Figures (4) and (5) we see that the higher harmonics are always relatively small and are only important when $a < 2.5$. Thus if they are to show up at all in the magnetic records, the majority of the trapped particle energy must reside on dipole shells for which $1.2 \leq a \leq 2.5$. The total external contribution to the magnetic field at the surface of the earth from storm time particles is shown in Figure (7) (after Slaucitajs and McNish, 1936). At low latitudes the horizontal component is seen to be essentially constant, and it is at low latitudes that we should expect the trapped particles to play the dominant role in establishing the storm time field. In Figure (5) we see that $I_t \propto H_t$ is essentially constant only when $1.7 \lesssim a \lesssim 2.0$. Thus it would appear that the majority of geomagnetically trapped particles which produced the magnetic storm field are trapped on dipole shells for which $1.3 \leq a \leq 2.5$ with a maximum on the dipole shell with $a \approx 1.7$.

To obtain an order of magnitude estimate of the particle energy density we assume that the equatorial particle energy density is uniform on the interval $1.6 \leq a \leq 2.3$ and the pitch angle distribution is modified isotropic. A typical value of the horizontal component of the magnetic disturbance at the equator is $10^{-3}$ gauss. Since $U^* = H_t B_o/\gamma \sum I_{ti}$, we have $U^* = 2 \times 10^{-6}$ ergs/cm$^3$. (The earth's magnetic field energy density at $a = 2$ is $2 \times 10^{-4}$ ergs/cm$^3$.) If we assume all the particles are 5 kev electrons, the flux density is $6 \times 10^{11}$ electrons/cm$^2$ sec. Since observational results are still somewhat unsettled, any conclusions that might be drawn from a comparison with our results would be of questionable value.

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In summary our principle results are the following. (1) Particle motion around the equatorial plane in the trapping region is the principal source of magnetic field due to trapped particles. (2) The magnetic field throughout the earth due to particles trapped on dipole shells for which $a > 2.5$ is essentially uniform. (3) It seems likely that the majority of particles responsible for magnetic storms are trapped on dipole shells for which $1.3 \leq a \leq 2.5$ with a maximum concentration of particles around the dipole shell with $a \approx 1.7$.

Author

## IV. APPENDIX A

### Numerical Integration of Magnetic Field Components

The integrals which represent the magnetic field components, in general, must be evaluated on an automatic digital computer. In Appendix A we rewrite (42), (43), (56), and (57) in a more tractable form for machine calculation.

First, in the case of the drift current integrals, Equations (42) and (43), we observe that the coefficient of $P_n^1(\cos\theta')$ in $b_n$ is an even function of $\theta'$. Thus $b_n$ is an odd function of $\theta'$ for even n and an even function for odd n. Because the limits of integration of the inner integral are symmetric about $\theta' = \pi/2$, only the terms with n odd contribute to the integral, a result that should be anticipated from the symmetry of the sources about the equatorial plane. A further consequence of the odd-even properties of the integrand is that it is necessary to integrate only on the interval $(\eta(a), \pi/2)$, whence doubling the result gives the value of the desired integral.

Let

$$D(a,\theta') = a^2 \sin^9\theta' \frac{(1+\cos^2\theta')}{(1+3\cos^2\theta')^2} \cos\alpha_{min}\left(1+\tfrac{1}{3}\cos^2\alpha_{min}\right) \tag{A1}$$

Then the components of the disturbance field due to the drift motion of the trapped particles are

$$H_r \cong -24\pi \frac{U^*}{B_0\gamma} \int_{a_1}^{a_2} \bar{\beta}(a)\, da \int_{\eta(a)}^{\pi/2} S_r \, d\theta' \tag{A2}$$

and

$$H_t \cong 24\pi \frac{U^*}{B_0\gamma} \int_{a_1}^{a_2} \bar{\beta}(a)\, da \int_{\eta(a)}^{\pi/2} S_t \, d\theta' \tag{A3}$$

where

$$S_r = D(a,\theta')\sum_{n=1,3,\dots}^{7} r^{n-1}\,\frac{P_n^1(\cos\theta')}{(a\sin^2\theta')^{n-1}}\,P_n(\cos\theta) \tag{A4}$$

and

$$S_t = D(a,\theta')\sum_{n=1,3,\dots}^{7} \frac{1}{n}\, r^{n-1}\,\frac{P_n^1(\cos\theta')}{(a\sin^2\theta')^{n-1}}\,P_n^1(\cos\theta) \tag{A5}$$

To evaluate the gyration integrals, Equations (56) and (57), we must first subdivide the trapping region. Referring to Figure (6) we see that the limits of the r' integration in region $R_2$ are $a_1 \sin^2\theta' \le r' \le a_2 \sin^2\theta'$, whereas in $R_1$ and $R_3$, $r_{atm} \le r' \le a_2 \sin^2\theta'$. The ranges of θ' in the regions $R_1$, $R_2$, and $R_3$ are respectively

$$\eta(a_2) \le \theta' \le \eta(a_1)$$

$$\eta(a_1) \le \theta' \le \pi - \eta(a_1)$$

$$\pi - \eta(a_1) \le \theta' \le \pi - \eta(a_2)$$

If we measure distances in units of earth radii, interchange integration and summation, and resort to arguments similar to those used to evaluate the drift current integrals, we have

$$H_r \cong \frac{8\pi^2}{3}\,\frac{U^*}{B_0\gamma}\int_{\eta(a_2)}^{\eta(a_1)} d\theta' \int_{r_{atm}}^{a_2\sin^2\theta'} T_r\, dr' + \frac{8\pi^2}{3}\,\frac{U^*}{B_0\gamma}\int_{\eta(a_1)}^{\pi/2} d\theta' \int_{a_1\sin^2\theta'}^{a_2\sin^2\theta'} T_r\, dr' \tag{A6}$$

and

$$H_t \cong -\frac{8\pi^2}{3}\,\frac{U^*}{B_0\gamma}\int_{\eta(a_2)}^{\eta(a_1)} d\theta' \int_{r_{atm}}^{a_2\sin^2\theta'} T_t\, dr' - \frac{8\pi^2}{3}\,\frac{U^*}{B_0\gamma}\int_{\eta(a_1)}^{\pi/2} d\theta' \int_{a_1\sin^2\theta'}^{a_2\sin^2\theta'} T_t\, dr' \tag{A7}$$

where

$$T_r = G(r', \theta') \sum_{n=1,3,\ldots}^{7} n \left(\frac{r}{r'}\right)^{n-1} P_n(\cos\theta') \, P_n(\cos\theta) \tag{A8}$$

and

$$T_t = G(r', \theta') \sum_{n=1,3,\ldots}^{7} n \left(\frac{r}{r'}\right)^{n-1} P_n(\cos\theta') \, P_n^1(\cos\theta) \tag{A9}$$

The iterated integrals are evaluated by a twofold application of Simpson's rule. Each iteration is done with k subdivisions of the interval and then with 2k subdivisions of the interval and the results compared. This process is repeated until successive estimates of the double integral agree to within one percent.

# V. APPENDIX B

## Solution of Poisson's Equation in Source Free Space

In Appendix B we suspend the convention of using primes to refer to source points and use primes instead to denote differentiation with respect to r. By writing the potential $\Omega$ and $\nabla \cdot \underline{M}$ in terms of Legendre polynomials, the problem of solving Poisson's equation

$$\nabla^2 \Omega = 4\pi \, \nabla \cdot \underline{M} \tag{B1}$$

in a source free region of space is reduced to the problem of solving an ordinary second order differential equation. The complete solution of (B1) is conveniently dealt with in two parts by construction a sphere centered at the origin with a radius $r = r_{atm}$. For the present we are interested only in the solution inside the sphere. Let

$$\nabla \cdot \underline{M} = \sum_{n=0}^{\infty} g_n(r) P_n(\cos\theta) \tag{B2}$$

and

$$\Omega = \sum_{n=0}^{\infty} h_n(r) P_n(\cos\theta) \tag{B3}$$

Inserting these expressions in (B1) and equating coefficients of $P_n$ (cos $\theta$) gives

$$h'' + \frac{2}{r} h_n' - \frac{n(n+1)}{r^2} h_n = 4\pi g_n \tag{B4}$$

A particular integral of the reduced equation is $r^n$. Now let $h_n = r^n \sigma(r)$ and substitute this in (B4). The result is

$$\sigma'' + \frac{2}{r}(n+1)\sigma' = \frac{4\pi g_n}{r^n} \tag{B5}$$

or

$$\left(\sigma' r^{2n+2}\right)' = 4\pi g_n r^{n+2} \tag{B6}$$

Thus the first integral is

$$\sigma' r^{2(n+1)} = -4\pi \int_r^\infty g_n(v)\, v^{n+2} dv + c_1 \tag{B7}$$

A quadrature gives

$$\sigma = 4\pi \int_r^\infty \frac{du}{u^{2(n+1)}} \int_u^\infty g_n(v)\, v^{n+2} dv - \frac{c_1}{2n+2} \frac{1}{r^{2n+1}} + c_2 \tag{B8}$$

On interchanging the order of integration and integrating once we have

$$\sigma = \frac{4\pi}{2n+1} \int_r^\infty g_n(v)\, v^{n+2}\left[\frac{1}{r^{2n+1}} - \frac{1}{v^{2n+1}}\right] dv - \frac{c_1}{2n+1} \frac{1}{r^{2n+1}} + c_2 \tag{B9}$$

and so

$$h_n = 4\pi r^{-n-1} \int_r^\infty g_n(v)\, v^{n+2} dv - \frac{4\pi r^n}{2n+1} \int_r^\infty g_n(v)\, v^{-n+1} dv + \frac{c_1}{2n+1} r^{-n-1} + c_2 r^n \tag{B10}$$

As long as $r < r_{atm}$, the integrals include all of the magnetic sources and are therefore constants. Now to satisfy the condition that the potential be bounded near the origin, we choose $c_1$ so that the first and third terms cancel. Furthermore, if we define the potential at the origin to be zero, then $c_2 = 0$ and

$$h_n = -\frac{4\pi r^n}{2n+1} \int_r^\infty g_n(v)\, v^{-n+1} dv \tag{B11}$$

Since

$$g_n = -\frac{2n+1}{2} \int_{\eta(a)}^{\pi-\eta(a)} \nabla\cdot \underline{M}\, P_n(\cos\theta) \sin\theta\, d\theta \tag{B12}$$

we have

$$h_n = 2\pi \int_{\eta(a)}^{\pi-\eta(a)} d\theta \int_{r_0}^\infty \frac{\nabla\cdot \underline{M}\, P_n(\cos\theta)}{r^{n-1}} \sin\theta\, dr \tag{B13}$$

for $r_0 \leq r_{atm}$.

# VI. TABLES

Table 1a. $I_r = H_r B_o/\gamma U^*$ for $10^o$ steps of colatitude $\theta$. $H_r$ is the radial component of magnetic field at the surface of the earth due to particles trapped on field lines whose equatorial radii are between a and a + 0.1 (measured in earth radii) for the range 1.3 a 4.0. The equatorial particle energy density is uniform and the pitch angle distribution is modified isotropic. See the text for definitions of $B_o$, $\gamma$, and $U^*$. The entries a E b are read as a x $10^b$, e.g., -1.48E 01 = $-1.48 \times 10^1$.

Table 1a.

| THETA | 0.0 | 10.0 | 20.0 | 30.0 | 40.0 | 50.0 | 60.0 | 70.0 | 80.0 | 90.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | | | | | | | | | | |
| 1.3 | -3.59E 00 | -3.63E 00 | -3.77E 00 | -3.95E 00 | -4.20E 00 | -4.64E 00 | -4.89E 00 | -5.51E 00 | -3.62E 00 | -0. |
| 1.4 | -5.99E 00 | -6.10E 00 | -6.40E 00 | -6.53E 00 | -6.80E 00 | -7.00E 00 | -7.02E 00 | -7.47E 00 | -4.47E 00 | -0. |
| 1.5 | -8.70E 00 | -8.83E 00 | -9.06E 00 | -9.20E 00 | -9.40E 00 | -9.11E 00 | -8.70E 00 | -9.05E 00 | -5.16E 00 | -0. |
| 1.6 | -1.16E 01 | -1.17E 01 | -1.17E 01 | -1.19E 01 | -1.18E 01 | -1.10E 01 | -1.02E 01 | -1.05E 01 | -5.16E 00 | -0. |
| 1.7 | -1.48E 01 | -1.48E 01 | -1.46E 01 | -1.47E 01 | -1.42E 01 | -1.29E 01 | -1.13E 01 | -1.13E 01 | -5.50E 00 | -0. |
| 1.8 | -1.80E 01 | -1.79E 01 | -1.77E 01 | -1.75E 01 | -1.67E 01 | -1.50E 01 | -1.24E 01 | -8.07E 00 | -5.93E 00 | -0. |
| 1.9 | -2.12E 01 | -2.11E 01 | -2.08E 01 | -2.04E 01 | -1.91E 01 | -1.68E 01 | -1.36E 01 | -8.29E 00 | -6.33E 00 | -0. |
| 2.0 | -2.46E 01 | -2.44E 01 | -2.41E 01 | -2.33E 01 | -2.16E 01 | -1.88E 01 | -1.48E 01 | -9.39E 00 | -6.63E 00 | -0. |
| 2.1 | -2.81E 01 | -2.79E 01 | -2.75E 01 | -2.63E 01 | -2.41E 01 | -2.08E 01 | -1.61E 01 | -1.06E 01 | -6.77E 00 | -0. |
| 2.2 | -3.18E 01 | -3.15E 01 | -3.09E 01 | -2.94E 01 | -2.68E 01 | -2.29E 01 | -1.75E 01 | -1.19E 01 | -6.58E 00 | -0. |
| 2.3 | -3.55E 01 | -3.52E 01 | -3.45E 01 | -3.26E 01 | -2.95E 01 | -2.51E 01 | -1.92E 01 | -1.31E 01 | -6.78E 00 | -0. |
| 2.4 | -3.94E 01 | -3.91E 01 | -3.81E 01 | -3.59E 01 | -3.26E 01 | -2.75E 01 | -2.10E 01 | -1.45E 01 | -7.46E 00 | -0. |
| 2.5 | -4.35E 01 | -4.31E 01 | -4.19E 01 | -3.92E 01 | -3.54E 01 | -2.99E 01 | -2.30E 01 | -1.59E 01 | -8.17E 00 | -0. |
| 2.6 | -4.77E 01 | -4.73E 01 | -4.59E 01 | -4.29E 01 | -3.85E 01 | -3.24E 01 | -2.51E 01 | -1.74E 01 | -8.95E 00 | -0. |
| 2.7 | -5.20E 01 | -5.15E 01 | -4.98E 01 | -4.64E 01 | -4.16E 01 | -3.50E 01 | -2.71E 01 | -1.87E 01 | -9.59E 00 | -0. |
| 2.8 | -5.65E 01 | -5.59E 01 | -5.39E 01 | -5.02E 01 | -4.50E 01 | -3.78E 01 | -2.94E 01 | -2.03E 01 | -1.04E 01 | -0. |
| 2.9 | -6.11E 01 | -6.04E 01 | -5.82E 01 | -5.41E 01 | -4.83E 01 | -4.06E 01 | -3.17E 01 | -2.19E 01 | -1.12E 01 | -0. |
| 3.0 | -6.58E 01 | -6.50E 01 | -6.25E 01 | -5.80E 01 | -5.18E 01 | -4.35E 01 | -3.39E 01 | -2.34E 01 | -1.20E 01 | -0. |
| 3.1 | -7.07E 01 | -6.99E 01 | -6.70E 01 | -6.22E 01 | -5.55E 01 | -4.66E 01 | -3.63E 01 | -2.51E 01 | -1.28E 01 | -0. |
| 3.2 | -7.57E 01 | -7.47E 01 | -7.17E 01 | -6.64E 01 | -5.91E 01 | -4.95E 01 | -3.88E 01 | -2.68E 01 | -1.36E 01 | -0. |
| 3.3 | -8.07E 01 | -7.97E 01 | -7.63E 01 | -7.08E 01 | -6.28E 01 | -5.26E 01 | -4.12E 01 | -2.84E 01 | -1.45E 01 | -0. |
| 3.4 | -8.62E 01 | -8.50E 01 | -8.14E 01 | -7.53E 01 | -6.69E 01 | -5.63E 01 | -4.40E 01 | -3.03E 01 | -1.55E 01 | -0. |
| 3.5 | -9.15E 01 | -9.03E 01 | -8.64E 01 | -7.99E 01 | -7.09E 01 | -5.96E 01 | -4.66E 01 | -3.21E 01 | -1.64E 01 | -0. |
| 3.6 | -9.72E 01 | -9.58E 01 | -9.17E 01 | -8.48E 01 | -7.53E 01 | -6.32E 01 | -4.95E 01 | -3.41E 01 | -1.73E 01 | -0. |
| 3.7 | -1.03E 02 | -1.01E 02 | -9.69E 01 | -8.96E 01 | -7.95E 01 | -6.68E 01 | -5.23E 01 | -3.59E 01 | -1.83E 01 | -0. |
| 3.8 | -1.09E 02 | -1.07E 02 | -1.02E 02 | -9.45E 01 | -8.37E 01 | -7.04E 01 | -5.51E 01 | -3.78E 01 | -1.93E 01 | -0. |
| 3.9 | -1.15E 02 | -1.13E 02 | -1.08E 02 | -9.98E 01 | -8.85E 01 | -7.45E 01 | -5.82E 01 | -3.98E 01 | -2.03E 01 | -0. |
| 4.0 | -1.21E 02 | -1.19E 02 | -1.14E 02 | -1.05E 02 | -9.32E 01 | -7.84E 01 | -6.13E 01 | -4.21E 01 | -2.14E 01 | -0. |

Table 1b. $I_t = H_t B_o / \gamma U^*$ for $10^o$ steps of colatitude $\theta$. $H_t$ is the transverse component of magnetic field at the surface of the earth due to particles trapped on field lines whose equatorial radii are between a and a + 0.1 (measured in earth radii) for the range 1.3 a 4.0. The equatorial particle energy density is uniform and the pitch angle distribution is modified isotropic. See the text for definitions of B., $\gamma$, and $U^*$. The entries a E b are read as a x $10^b$, e.g., -1.48E 01 = -1.48 x $10^1$.

Table 1b.

| THETA | 0.0 | 10.0 | 20.0 | 30.0 | 40.0 | 50.0 | 60.0 | 70.0 | 80.0 | 90.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| A | | | | | | | | | | |
| 1.3 | 0. | 2.93E-01 | 6.77E-01 | 1.04E 00 | 1.76E 00 | 2.49E 00 | 3.99E 00 | 6.43E 00 | 9.53E 00 | 1.08E 01 |
| 1.4 | 0. | 4.94E-01 | 1.28E 00 | 2.15E 00 | 3.06E 00 | 4.54E 00 | 7.29E 00 | 9.74E 00 | 1.31E 01 | 1.53E 01 |
| 1.5 | 0. | 8.34E-01 | 1.92E 00 | 3.38E 00 | 4.63E 00 | 7.06E 00 | 1.05E 01 | 1.28E 01 | 1.43E 01 | 1.94E 01 |
| 1.6 | 0. | 1.26E 00 | 2.66E 00 | 4.70E 00 | 6.58E 00 | 9.83E 00 | 1.33E 01 | 1.55E 01 | 1.60E 01 | 1.76E 01 |
| 1.7 | 0. | 1.73E 00 | 3.57E 00 | 5.81E 00 | 8.75E 00 | 1.28E 01 | 1.60E 01 | 1.81E 01 | 1.83E 01 | 1.79E 01 |
| 1.8 | 0. | 2.24E 00 | 4.60E 00 | 7.43E 00 | 1.11E 01 | 1.57E 01 | 1.87E 01 | 2.08E 01 | 2.11E 01 | 2.10E 01 |
| 1.9 | 0. | 2.77E 00 | 5.73E 00 | 9.21E 00 | 1.36E 01 | 1.82E 01 | 2.14E 01 | 2.36E 01 | 2.38E 01 | 2.42E 01 |
| 2.0 | 0. | 3.40E 00 | 6.96E 00 | 1.11E 01 | 1.62E 01 | 2.08E 01 | 2.41E 01 | 2.63E 01 | 2.71E 01 | 2.75E 01 |
| 2.1 | 0. | 4.05E 00 | 8.28E 00 | 1.32E 01 | 1.87E 01 | 2.34E 01 | 2.70E 01 | 2.93E 01 | 3.04E 01 | 3.09E 01 |
| 2.2 | 0. | 4.73E 00 | 9.68E 00 | 1.52E 01 | 2.12E 01 | 2.61E 01 | 2.99E 01 | 3.23E 01 | 3.39E 01 | 3.45E 01 |
| 2.3 | 0. | 5.46E 00 | 1.11E 01 | 1.73E 01 | 2.37E 01 | 2.89E 01 | 3.30E 01 | 3.57E 01 | 3.74E 01 | 3.81E 01 |
| 2.4 | 0. | 6.21E 00 | 1.26E 01 | 1.94E 01 | 2.62E 01 | 3.18E 01 | 3.64E 01 | 3.92E 01 | 4.12E 01 | 4.19E 01 |
| 2.5 | 0. | 6.97E 00 | 1.41E 01 | 2.16E 01 | 2.87E 01 | 3.48E 01 | 3.97E 01 | 4.29E 01 | 4.51E 01 | 4.59E 01 |
| 2.6 | 0. | 7.76E 00 | 1.57E 01 | 2.38E 01 | 3.13E 01 | 3.79E 01 | 4.31E 01 | 4.67E 01 | 4.92E 01 | 5.01E 01 |
| 2.7 | 0. | 8.57E 00 | 1.72E 01 | 2.60E 01 | 3.42E 01 | 4.11E 01 | 4.66E 01 | 5.06E 01 | 5.32E 01 | 5.42E 01 |
| 2.8 | 0. | 9.40E 00 | 1.88E 01 | 2.83E 01 | 3.69E 01 | 4.44E 01 | 5.04E 01 | 5.47E 01 | 5.76E 01 | 5.87E 01 |
| 2.9 | 0. | 1.02E 01 | 2.05E 01 | 3.06E 01 | 3.98E 01 | 4.78E 01 | 5.42E 01 | 5.89E 01 | 6.20E 01 | 6.32E 01 |
| 3.0 | 0. | 1.11E 01 | 2.21E 01 | 3.30E 01 | 4.27E 01 | 5.13E 01 | 5.81E 01 | 6.32E 01 | 6.65E 01 | 6.77E 01 |
| 3.1 | 0. | 1.19E 01 | 2.39E 01 | 3.54E 01 | 4.57E 01 | 5.50E 01 | 6.23E 01 | 6.77E 01 | 7.13E 01 | 7.26E 01 |
| 3.2 | 0. | 1.28E 01 | 2.56E 01 | 3.79E 01 | 4.90E 01 | 5.87E 01 | 6.65E 01 | 7.23E 01 | 7.61E 01 | 7.73E 01 |
| 3.3 | 0. | 1.37E 01 | 2.74E 01 | 4.04E 01 | 5.22E 01 | 6.25E 01 | 7.05E 01 | 7.69E 01 | 8.09E 01 | 8.22E 01 |
| 3.4 | 0. | 1.47E 01 | 2.92E 01 | 4.31E 01 | 5.56E 01 | 6.65E 01 | 7.53E 01 | 8.21E 01 | 8.64E 01 | 8.78E 01 |
| 3.5 | 0. | 1.57E 01 | 3.11E 01 | 4.58E 01 | 5.90E 01 | 7.05E 01 | 7.99E 01 | 8.71E 01 | 9.15E 01 | 9.30E 01 |
| 3.6 | 0. | 1.66E 01 | 3.30E 01 | 4.86E 01 | 6.26E 01 | 7.49E 01 | 8.49E 01 | 9.24E 01 | 9.71E 01 | 9.87E 01 |
| 3.7 | 0. | 1.76E 01 | 3.50E 01 | 5.13E 01 | 6.62E 01 | 7.91E 01 | 8.96E 01 | 9.75E 01 | 1.02E 02 | 1.04E 02 |
| 3.8 | 0. | 1.87E 01 | 3.70E 01 | 5.42E 01 | 6.99E 01 | 8.36E 01 | 9.44E 01 | 1.03E 02 | 1.08E 02 | 1.10E 02 |
| 3.9 | 0. | 1.97E 01 | 3.90E 01 | 5.72E 01 | 7.37E 01 | 8.81E 01 | 9.99E 01 | 1.09E 02 | 1.14E 02 | 1.16E 02 |
| 4.0 | 0. | 2.08E 01 | 4.12E 01 | 6.03E 01 | 7.77E 01 | 9.28E 01 | 1.05E 02 | 1.14E 02 | 1.20E 02 | 1.22E 02 |

# VII. FIGURES

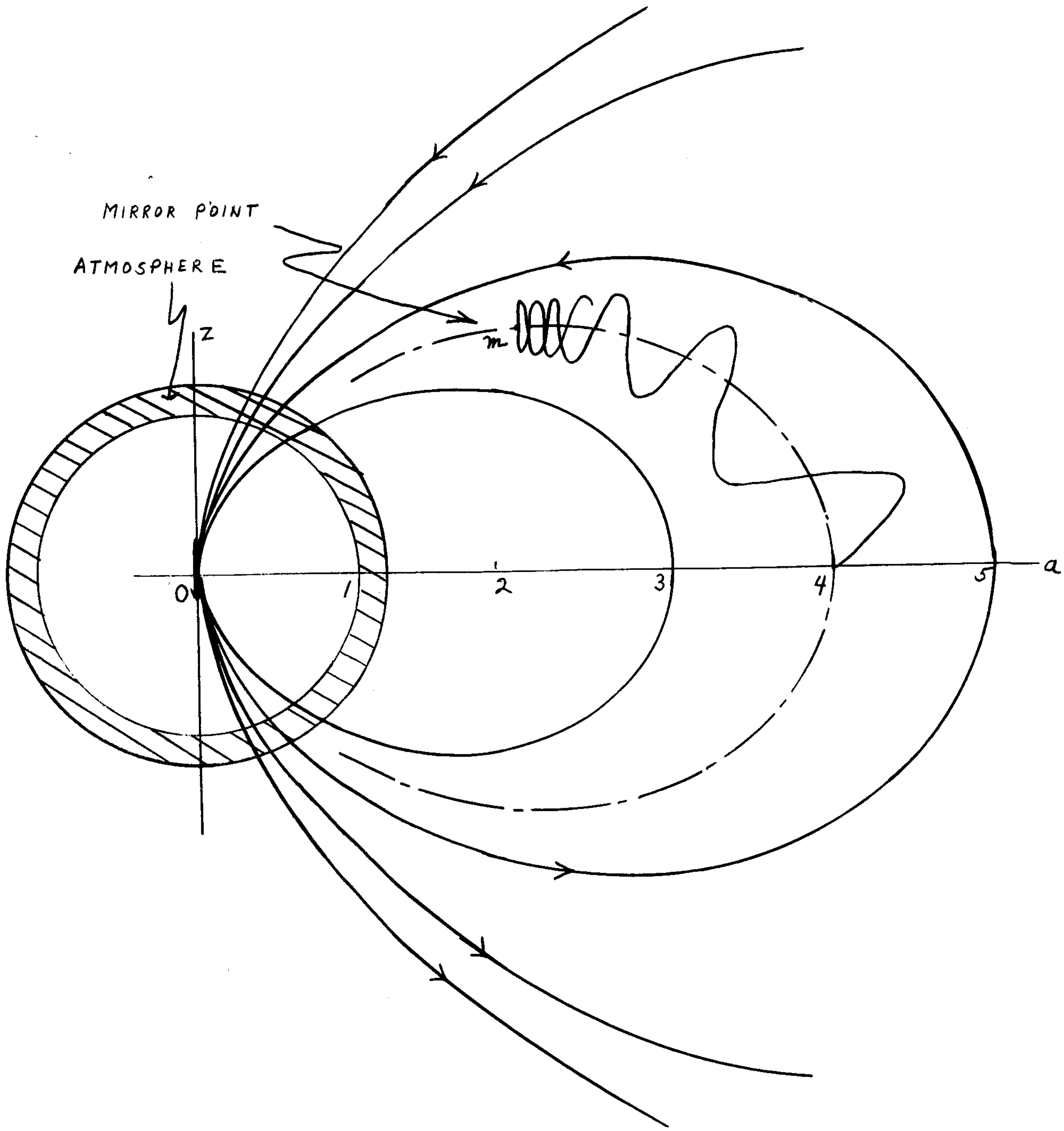



Figure 1. Geometry of the idealized magnetic field of the earth and the helical trajectory of a trapped particle. The geomagnetic north pole points in the negative z direction.

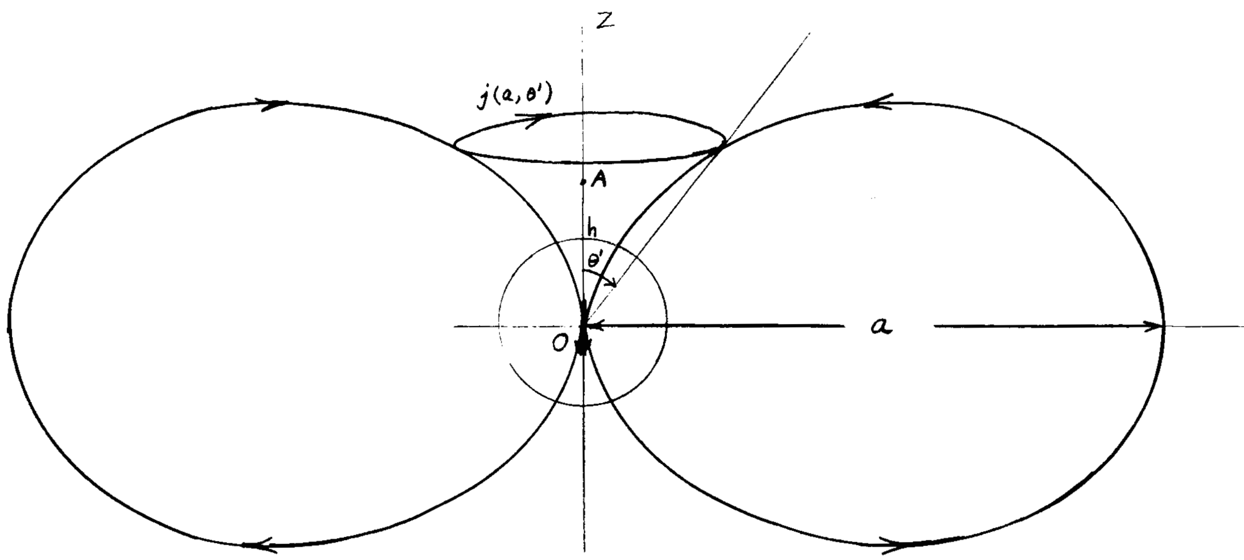



Figure 2. An element of ring current of density $j(a, \theta')$ moving westward around the earth. The geomagnetic north pole points in the negative z direction.

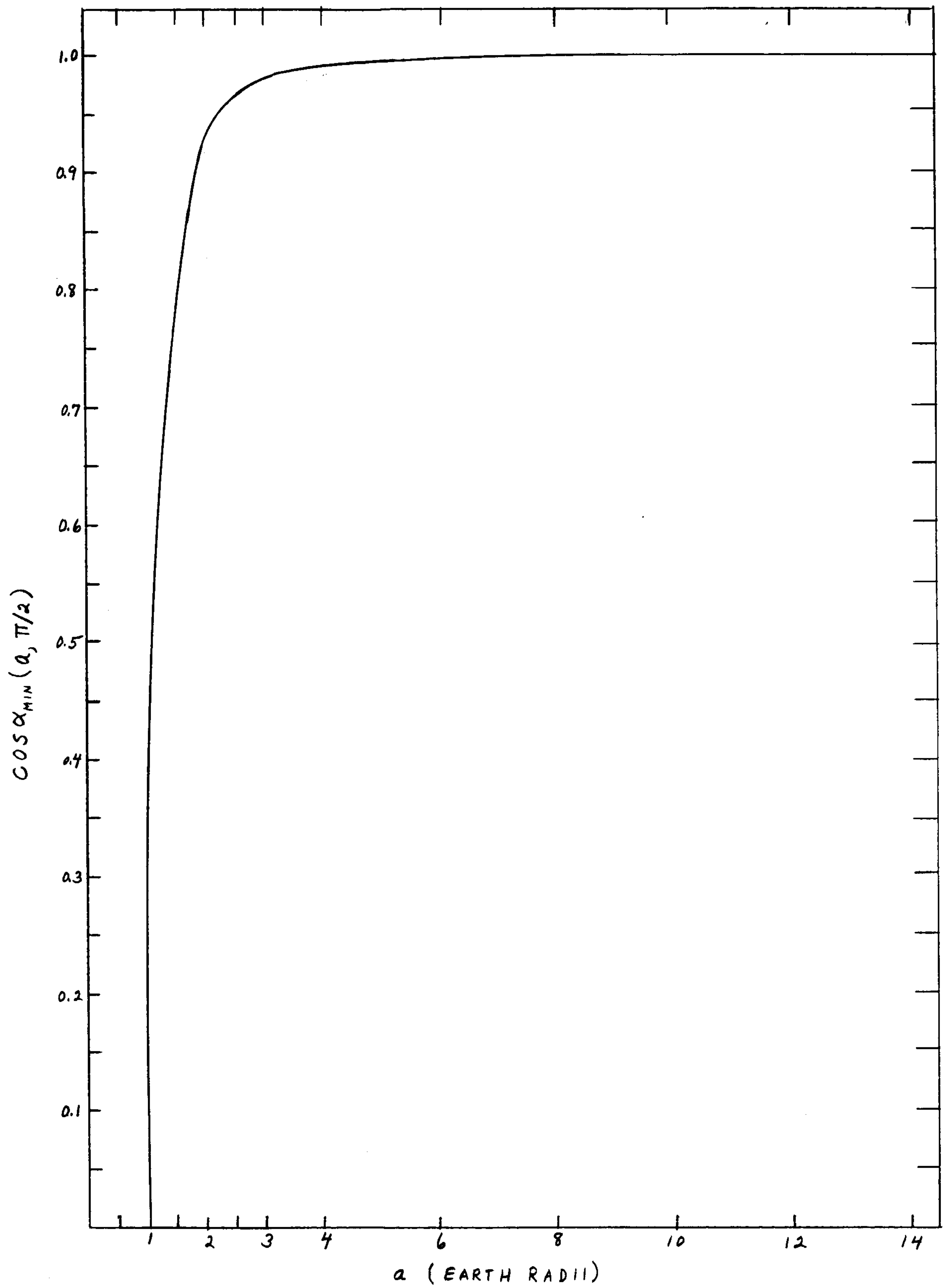
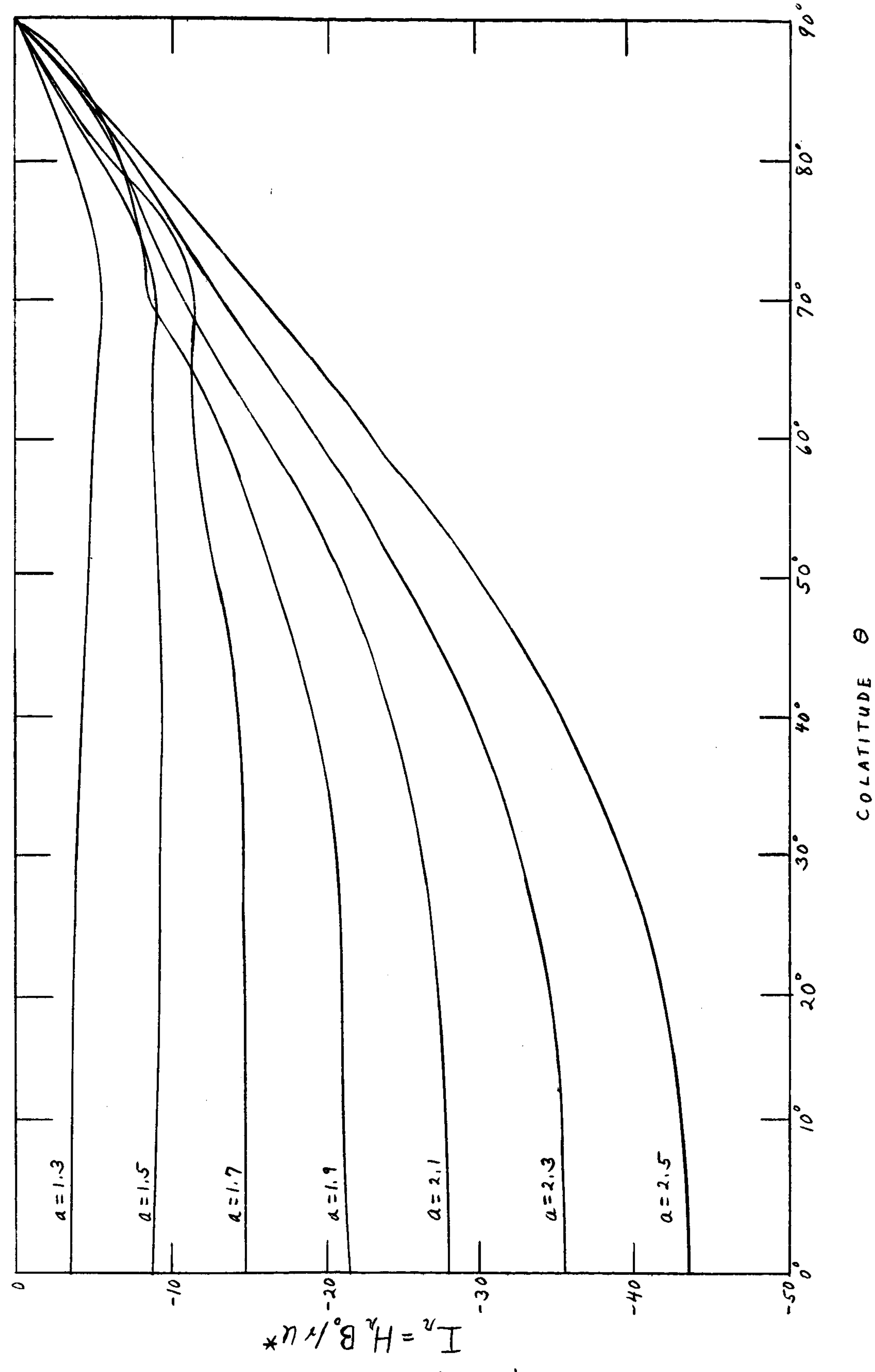



Figure 3. Cosine of the equatorial pitch angle of particles which mirror at the top of the atmosphere (1200 km) as a function of the equatorial radius of the dipole shell on which the particles reside.

Figure 4. $I_r = H_r B_o/\gamma U^*$ as a function of colatitude $\theta$ where $H_r$ is the radial component of the magnetic field at the surface of the earth due to particles trapped on field lines whose equatorial radii (measured in earth radii) are between a and a + 0.1. $I_r$ is plotted for $1.3 \leq a \leq 2.5$ in steps of 0.2. The equatorial particle energy density is uniform and the pitch angle distribution is modified isotropic. $B_o$, $\gamma$, and $U^*$ are defined in the text.



a = 1.3
a = 1.5
a = 1.7
a = 1.9
a = 2.1
a = 2.3
a = 2.5
0
-10
-20
-30
-40
-50
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
COLATITUDE θ
$I_r = H_r B_o / r u*$


Figure 4.

Figure 5. $I_t = B_o/\gamma U^*$ as a function of colatitude $\theta$ where $H_t$ is the transverse component of magnetic field at the surface of the earth due to particles trapped on field lines whose equatorial radii (measure in earth radii) are between a and a + 0.1. $I_t$ is plotted for $1.3 \leq a \leq 2.5$ in steps of 0.2. The equatorial particle energy density is uniform and the pitch angle distribution is modified isotropic. $B_o$, $\gamma$, and $U^*$ are defined in the text.

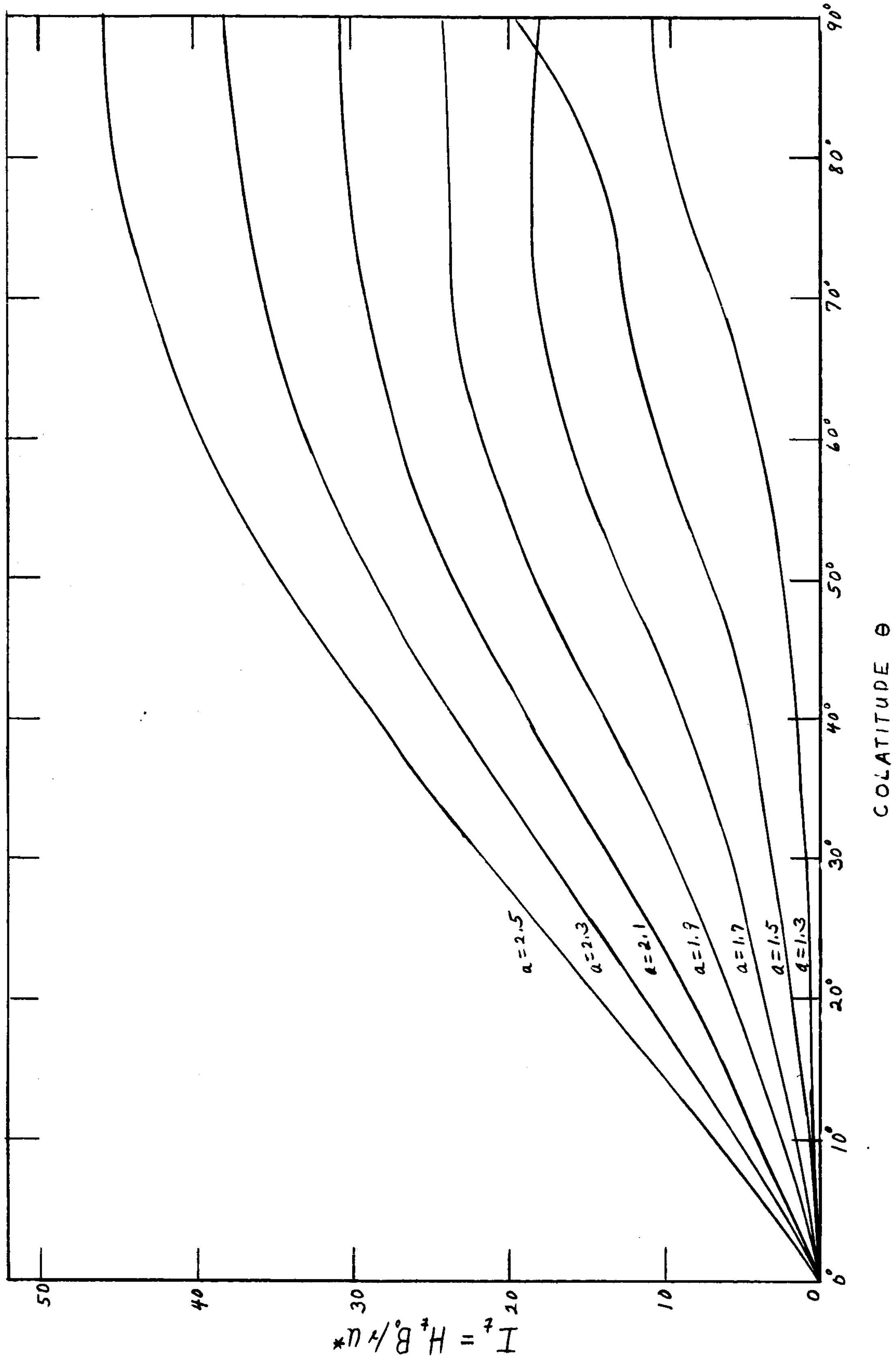

COLATITUDE θ
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
0
10
20
30
40
50
a = 2.5
a = 2.3
a = 2.1
a = 1.9
a = 1.7
a = 1.5
a = 1.3

Figure 5

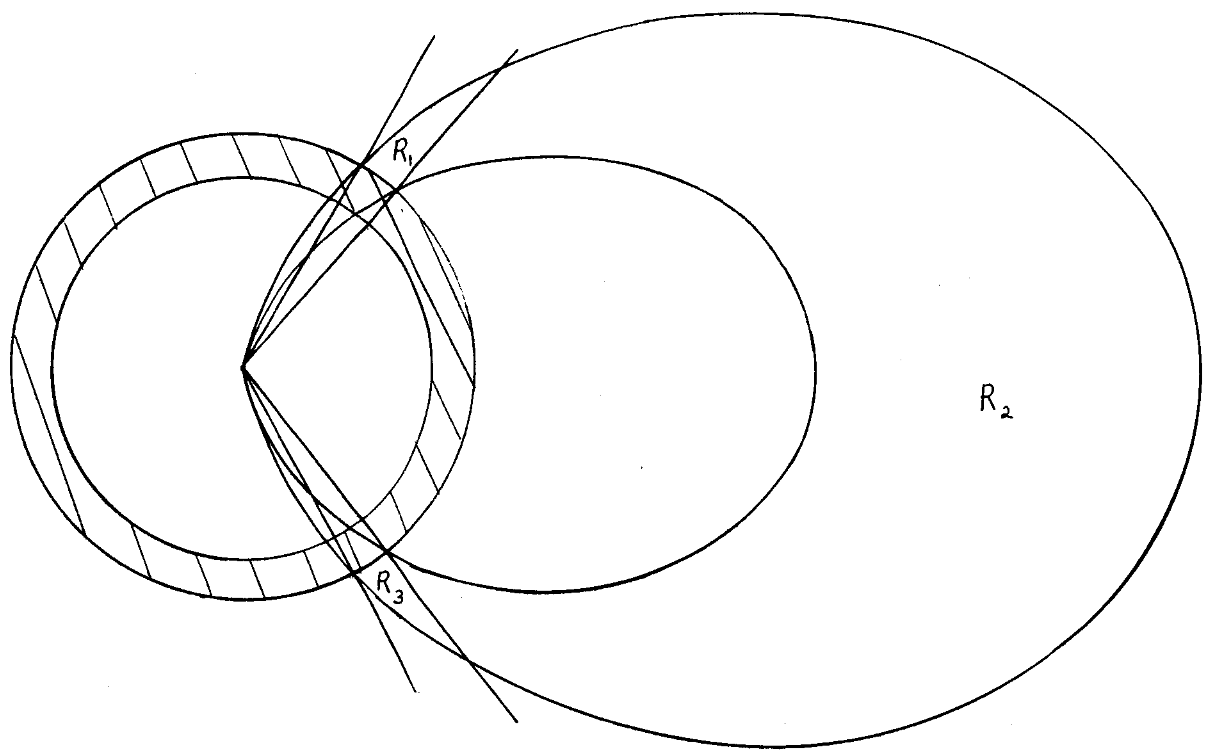



Figure 6. Subdivision of the trapping region for integration purposes.

Figure 7. Total external contribution to the average differences in magnetic intensity for disturbed minus quiet days, 1927 (after Slaucitajs and McNish, 1936).

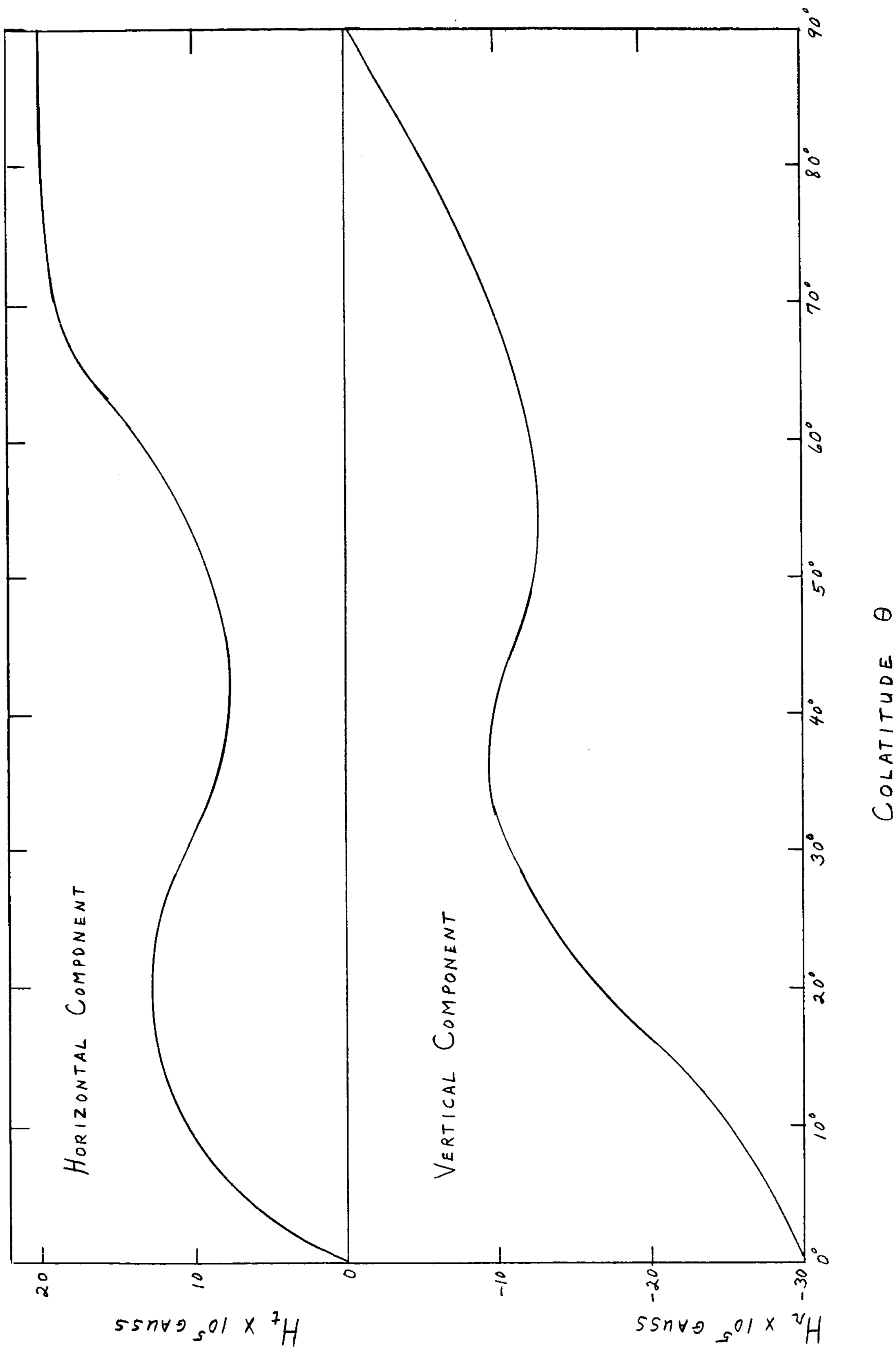

HORIZONTAL COMPONENT
VERTICAL COMPONENT
COLATITUDE θ
H_t x 10^5 GAUSS
H_r x 10^5 GAUSS
20
10
0
-10
-20
-30
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°

Figure 7

## VIII. BIBLIOGRAPHY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